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Stability of Charged Solitons

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The stability of charged solitons described by the relativistic complex scalar field is investigated by the direct Lyapunov method. It is shown that the stability of pulson-type solitons can only be conditional. Some necessary and sufficient conditions for the stability of stationary solitons with fixed charge are established. Several examples are considered.

In recent years the interest towards regular or particlelike solutions to nonlinear field equations has increased considerably. These solutions, called solitons, find numerous applications in plasma physics, nonlinear optics, the theory of elementary particles etc. (Makhankov, 1978; Faddeev and Korepin, 1978). In this connection, very often, the stability of solitons with respect to small initial perturbations acquires special importance. Although the methods for the investigation of stability are well developed (Zubov, 1957), their practical application is often connected with serious mathematical difficulties.

In the present paper some conditions for the stability of charged solitons described by the complex scalar field $\varphi(t,\mathbf{x}): R^1\times R^3\to\mathbb{C}^1$, satisfying the natural boundary condition $\lim_{|\mathbf{x}|\to\infty}\varphi(x)=0$, $x\equiv(t,\mathbf{x})$, are established.

Let the Lagrangian density have the Lorentz-invariant form:

$$L=-\tfrac{1}{2}F(p,s);\qquad p=-\partial_\mu\varphi^*\partial^\mu\varphi;\qquad s=|\varphi|^2 \tag{1}$$

and the corresponding field equations have the stationary regular solution

$$\varphi_0(x)=u(\mathbf{x})\exp(-i\omega t);\qquad u^*=u;\qquad \omega=\text{const} \tag{2}$$

describing the charged soliton at rest. If U denotes the set of functions obtained from φ_0 by means of 3-translations, 3-rotations, and gauge

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transformations $\varphi_0 \rightarrow \varphi_0 \exp(i\alpha), \alpha = \text{const}$, then, by definition, the function $\varphi(x) \notin U$ describes the perturbed soliton.

Let $\phi(x) \equiv \varphi(x)\exp(i\omega t)$. Following the papers by Movchan (1960) and Slobodkin (1964), we introduce the metrics (distances) $\rho_0[\xi^0]$ and $\rho[\xi]$ for the characterization of the initial perturbation $\xi^0 \equiv \phi(0,\mathbf{x}) - u(\mathbf{x})$ and the current perturbation $\xi \equiv \phi(x) - u(\mathbf{x})$, respectively. Putting $\xi = \xi_1 + i\xi_2$, $\xi_i^* = \xi_i$, and denoting the norms in $L_2(R^3)$ and Sobolev space $W_2^1(R^3)$ by $\|\cdot\|$ and $\|\cdot\|'$ respectively, we choose the metrics ρ_0, ρ in the form

$$\rho_0[\xi^0] = \sum_{i=1}^{2} \{\|\dot{\xi}_i^0\| + \|\xi_i^0\|'\}, \qquad \rho[\xi] = \inf_{u \in U} \sum_{i=1}^{2} \|\xi_i\|$$

where $\dot{\xi}_i \equiv \partial_0 \xi_i$.

Definition. The regular solution φ_0 is said to be stable in the Lyapunov sense with respect to the metrics ρ_0, ρ, if for each $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that from $\rho_0[\xi^0] < \delta$ it follows that $\rho[\xi] < \varepsilon$ for any $t > 0$.

Now let us consider a lemma of variational calculus that will be useful afterwards. Let the functional

$$V[\phi] = \int d^3x\, v(\phi, \nabla\phi)$$

be defined in the class of sectionally smooth functions $\phi(\mathbf{x}): R^3 \rightarrow R^n$, $\phi(\infty) = 0$. Let $u(\alpha; \mathbf{x})$ be the family of its extremal fields given by the parameters $\alpha = \{\alpha_i\}$.

> *Lemma.* If there exist the constants C_i, not all equal to zero, such that $f \equiv \Sigma_i C_i(\partial u / \partial \alpha_i)_{\alpha=0} = 0$ on some surface S separating in R^3 a domain Ω of nonzero measure, then $\delta^2 V$ is sign changing in the neighborhood of $u(0; \mathbf{x})$.

Proof. Consider the perturbation $\delta\phi$ such that $\delta\phi = 0$ for $\mathbf{x} \in \Omega$ and $\delta\phi = f$ for $\mathbf{x} \in \Omega$. Then due to the properties of extremal fields $\delta^2 V[\delta\phi] = 0$. However, the perturbation $\delta\phi$ is not the extremal point of the functional $\delta^2 V$, as it violates the Weierstrass–Erdmann matching condition. Hence, $\delta^2 V$ is sign changing.

Sometimes the regular solutions

$$\varphi_0 = u(t, \mathbf{x})\exp[-i\psi(t)], \qquad u^* = u \tag{3}$$

which are more general than (2), are considered.

Theorem 1. The regular solutions (3) are unstable, in the Lyapunov sense, in any model (1) (Rybakov, 1978).

Proof. According to the general theorem of stability (Zubov, 1957; Movchan, 1960) the motion φ_0 of the autonomous dynamical system (1) is stable with respect to the metrics ρ_0, ρ, if and only if in some neighborhood D of φ_0 there exists the Lyapunov's functional $V[\varphi]$ such that (i) it does not increase along the trajectories of the system, (ii) it is continuous with respect to the metric ρ_0, and (iii) it is positive definite with respect to the metric ρ. Thus if we suppose φ_0 to be stable, then there must exist a positive definite functional $V[\varphi]$ for which φ_0 is the extremal field. Since the model (1) is invariant under 3-translations $\mathbf{x}\to\mathbf{x}+\mathbf{a}$, $\mathbf{a}=\text{const}$, then $\varphi_0(t,\mathbf{x}+\mathbf{a})$ is also the extremal field for $V[\varphi]$. However, the equation $\partial_x u=0$ can be satisfied on some surface S because $u(x)$ is regular and $u(t,\infty)=0$. Hence, according to the lemma $\delta^2 V$ is sign changing, which contradicts the stability of φ_0 and proves the theorem.

From Theorem 1 it follows that only conditional stability of the regular solutions (2) and (3) can be achieved. One of the simplest conditions that can be imposed on the initial perturbations ξ^0 is the condition of charge fixation:

$$Q=\frac{i}{2}\int d^3x\, F_p(\varphi^*\partial_0\varphi-\partial_0\varphi^*\varphi)=Q[\varphi_0]\equiv Q_0 \tag{4}$$

Following (Makhankov, 1978), the stability under the condition (4) will be called Q stability.

Theorem 2. Regular nodal solutions (3) are Q unstable in any model (1).

Proof. As $u(t,\mathbf{x})=0$ on the nodal surface, all the conditions of the lemma are fulfilled for the family of extremal fields $\varphi_0\exp(i\alpha)$ allowed by the model (1). So the proof of Theorem 1 can be extended to this case too, the condition (4) being satisfied by assuming $\dot{\xi}_1^0=\dot{\xi}_2^0=0$ and $\|\xi_1^0\|\ll\|\xi_2^0\|$.

Now let us establish some sufficient conditions for Q stability of the nonnodal solutions (2) by choosing the Lyapunov's functional V in the form

$$V=E-\omega Q \tag{5}$$

where E is the field energy. It is clear that the fields (2) are extremal ones for the functional (5). The second variation of V can be written as

$$\delta^2 V = (\dot{\xi}_1, F_p \dot{\xi}_1) + (\dot{\xi}_2, (F_p - 2F_{pp}\omega^2 s)\dot{\xi}_2) + \sum_{i=1}^{2} (\xi_i, \hat{L}_i \xi_i) \tag{6}$$

where $(\cdot\ ,\ \cdot)$ denotes the scalar product in $L_2(R^3)$ and the Hermitian operators $\hat{L}_i$, $i = 1, 2$, have the form

$$\hat{L}_1 = \hat{L}_2 - 2\mathrm{div}[F_{pp} \nabla u (\nabla u \nabla)] + \mathrm{div}(F_{pp}\omega^2 - F_{ps}) \nabla \dot{s} + 2s(F_{ss} - 2\omega^2 F_{ps} + \omega^4 F_{pp})$$

$$\hat{L}_2 = -\mathrm{div}\, F_p \nabla - \omega^2 F_p + F_s$$

It is clear from (6) that if $\delta^2 V$ is positive definite, then

$$F_p > 0, \qquad h \equiv F_p - 2F_{pp}\omega^2 s > 0, \qquad F_p + 2F_{pp}(\nabla u)^2 > 0$$

In the linear approximation with respect to ξ, the condition (4) can be written as

$$(hu, \dot{\xi}_2) = 2\omega(g, \xi_1) \tag{7}$$

where

$$g \equiv -\mathrm{div}(F_{pp} s \nabla u) + u(F_p - \omega^2 s F_{pp} + s F_{ps})$$

From (7), using Schwartz's inequality, we get

$$(\dot{\xi}_2, h\dot{\xi}_2) \geqslant 4\omega^2 (g, \xi_1)^2 (u, hu)^{-1} \tag{8}$$

From (6) and (8) we have the estimate

$$\delta^2 V \geqslant (\dot{\xi}_1, F_p \dot{\xi}_1) + (\xi_2, \hat{L}_2 \xi_2) + (\xi_1, \hat{K}\xi_1) \equiv W[\dot{\xi}_1, \xi] \tag{9}$$

where

$$\hat{K}\xi_1 = \hat{L}_1 \xi_1 + 4\omega^2 g(g; \xi_1)(u, hu)^{-1}$$

Now we shall find the conditions for W to be positive definite with respect to the metric ρ, which can be enlarged by including $\|\dot{\xi}_1\|$. Note that due to the field equations we have $\hat{L}_2 u = 0$ and, therefore, according to Courant's theorem about the positivity of the first eigenfunction (Courant and Hilbert, 1953), the spectrum of $\hat{L}_2$ will be positive since $u > 0$. The zero mode is excluded here according to the definition of ρ, because for $\xi_2 = u \in U$, $\rho[u] = 0$.

Further, for the spectrum of $\hat{K}$ to be positive, it is necessary that $\hat{L}_1$ have not more than one negative eigenvalue, because in the opposite case $(g,\xi_1)=0$ can always be attained for $(\xi_1,\hat{L}_1\xi_1)<0$.

Let $\lambda(\omega)$ be the first eigenvalue of $\hat{K}$. According to Theorem 1, $\lambda(0)$ is always negative. Let us find the critical frequency ω_0 for which $\lambda(\omega_0)=0$ and which defines the boundary of the domain of Q stability $\omega>\omega_0$, if $\lambda_\omega \geqslant 0$ (owing to the symmetry $\omega\to-\omega$, it is sufficient to consider $\omega>0$). As sgn $\min_{\rho=\varepsilon}\delta^2V=\mathrm{sgn}(\omega-\omega_0)$, $u(\omega_0)$ is the saddle point of V with the curve of descent $u(\omega)$. Thus, for $\omega=\omega_0$, $\min_{\rho=\varepsilon}\delta^2V=\min_{\rho=\varepsilon}(\xi_1,\hat{K}\xi_1)=0$ and is achieved when $\xi_1=u_\omega$. Hence $\hat{K}u_\omega=0$, which along with $\hat{L}_2u=0$ leads to the following equation for ω_0 (Zastavenko, 1965; Rybakov, 1966b):

$$Q_{0\omega}\equiv\frac{d}{d\omega}\left(\omega\int d^3xF_ps\right)=0 \tag{10}$$

If $\omega>\omega_0$, then $(u_\omega,\hat{K}u_\omega)>0$ or, $Q_{0\omega}(Q_{0\omega}-(u,hu))>0$, whence with the help of (10) we get the inequality to determine the domain of Q stability (Vakhitov and Kolokolov, 1973; Friedberg et al., 1976):

$$Q_{0\omega}<0 \tag{11}$$

Note, once again, that the zero modes of the type $\xi_i=C_i\partial_iu$ are excluded according to the definition of the metric ρ.

Thus we come to the following conclusion.

> *Theorem 3.* Nonnodal regular solutions (2), in the model (1), are Q stable and the domain of Q stability is determined by the inequality (11), if the following conditions hold: (a) $\lambda_\omega\geqslant 0$; (b) the operator $\hat{L}_1$ has only one negative eigenvalue.

Let us see some examples.

1) $F=p+s-s^n/n$.

In this case (Jidkov and Shirikov, 1964), for $1<n<3$ and $|\omega|<1$, there exist spherically symmetric regular solutions (2). Changing the variables

$$|\mathbf{x}|=\rho(1-\omega^2)^{1/2},\qquad u=v(1-\omega^2)^{1/2(n-1)} \tag{12}$$

we get

$$Q_0(\omega)=\mathrm{const}\,\omega(1-\omega^2)^{(5-3n)/2(n-1)}$$

Therefore (11) is fulfilled only if $n<5/3$. Let $n=3/2$. This case was investigated in (Synge, 1961). The energy of the nonnodal solution is

$E_0 = 4\pi(1+2\omega^2)(1-\omega^2)^{1/2}3.4744\ldots$ Operator $\hat{K}$ has the following structure:

$$\hat{K} = \hat{L}_1 + 4\omega^2 P_u, \qquad \hat{L}_1 = -\Delta - \omega^2 + 1 - 2u$$

P_u being the projector on $u/\|u\|$. Using (12), the equation $\hat{K}\psi = \lambda\psi$ can be written in the form

$$[-\Delta + 1 - 2v(\rho) + \alpha(\omega) P_v]\psi = \tilde{\lambda}\psi \tag{13}$$

where $\alpha(\omega) = 4\omega^2(1-\omega^2)^{-1}$, $\tilde{\lambda} = \lambda(1-\omega^2)^{-1}$. Differentiating (13) with respect to α, we get $\tilde{\lambda}_\alpha = (\psi, P_v\psi)$ and, therefore, $\tilde{\lambda}_\omega \geqslant 0$, that replaces the condition (a) of Theorem 3 since $\operatorname{sgn}\tilde{\lambda} = \operatorname{sgn}\lambda$.

Now let us verify that the operator $\hat{L}_1$ has only one negative eigenvalue. As from (12), $\hat{L}_1 = (1-\omega^2)(-\Delta + 1 - 2v)$, consider the equation

$$[-\Delta + 1 - 2v(\rho)]\chi = \nu\chi \tag{14}$$

Separating the angular variables in (14): $\chi_{lk} = R_1(\rho) Y_{lk}$, we come to the conclusion that for $l = 1$, $R_1 = dv/d\rho$ and $\nu = 0$. Further, as $v(\rho)$ is monotonic (Coleman et al., 1978), R_1 has no internal zeros and hence, according to Courant's theorem, $\nu = 0$ is the lowest eigenvalue for $l \neq 0$. Therefore ν is negative only for s states. According to Sturm's theorem (Tricomi, 1961) the number of s states with $\nu < 0$ is equal to the number of internal zeros of the solution to the equation

$$[-d^2/d\rho^2 + 1 - 2v(\rho)]y(\rho) = 0 \tag{15}$$

if $y(0) = 0$, $y'(0) = 1$. Numerical calculations show that $y(\rho)$ has only one internal zero for $\rho \approx 1.32$. Thus we see that all conditions of Theorem 3 are fulfilled. The domain of Q stability is $1 > |\omega| > 2^{-1/2}$.

2) $F = p + s(m^2 + 1 - \ln s)$; $m = \text{const}$.

The model, introduced in (Bialynicki–Birula and Mycielski, 1975), has the simple solution of the type (2):

$$u = \exp\tfrac{1}{2}(3 + m^2 - \omega^2 - r^2); \qquad r = |\mathbf{x}|, \quad \forall\omega \in R$$

with the energy $E_0 = \frac{1}{2}\pi^{3/2}(1+2\omega^2)\exp(3 + m^2 - \omega^2)$ and charge $Q_0(\omega) = \text{const}\,\omega\exp(-\omega^2)$. So, the condition (11) is satisfied for $|\omega| > 2^{-1/2}$. Operator $\hat{K}$ has the form

$$\hat{K} = -\Delta + r^2 - 5 + 4\omega^2 P_u$$

Its spectrum can be found immediately:

$$\lambda_0 = -2 + 4\omega^2, \qquad \lambda_n = 2(n-1), \qquad n = 1, 2, \ldots$$

Thus the domain of Q stability is $|\omega| > 2^{-1/2}$.

3) $F = p + m^2 s + s(\frac{3}{2}\ln s)^{4/3} + 2\int_0^s dx(\frac{3}{2}\ln x)^{1/3}$.

The model has the solution (2) of the form $u = \exp(-r^3/3)$; $|\omega| = m$, with the energy $E_0 = 4\pi[(m^2/2) + \Gamma(\frac{1}{3})/9]$. Operator $\hat{K}$ has the form

$$\hat{K} = -\Delta + r^4 - 8r + 4/r^2 + 4m^2 P_u$$

Differentiating the equation $\hat{K}\psi = \lambda\psi$ with respect to m^2, we get

$$d\lambda/dm^2 = 4(\psi, P_u\psi) \geqslant 0$$

It can directly be checked that $\psi = r^2 u$ is the eigenfunction of $\hat{K}$ for $\lambda = 0$, if

$$m^2 = m_0^2 \equiv \frac{\int_0^\infty r^2 u^2 dr}{2\int_0^\infty r^4 u^2 dr} = \frac{\left(\frac{2}{3}\right)^{2/3}}{2\Gamma\left(\frac{5}{3}\right)}$$

Therefore, the domain of Q stability is $m^2 > m_0^2$ if the operator $\hat{L}_1 = \hat{K}(m=0)$ has only one negative eigenvalue. The verification of this point is the same as in example 1). The equation analogous to (15) is

$$(-d^2/dr^2 + r^4 - 8r + 4/r^2)y = 0$$

Its solution with the property $y(0) = 0$ is expressed through a confluent hypergeometric function:

$$y = r^\sigma \exp(-r^3/3) F\left[\tfrac{1}{3}(\sigma - 2), \tfrac{1}{3}(\sigma + 2), \tfrac{2}{3}r^3\right], \qquad \sigma = \tfrac{1}{2}\left[(17)^{1/2} - 1\right]$$

and has only one internal zero.

Concluding, it should be noted that the scalar field models considered here have just the illustrative character. However, in more realistic models dealing with vector and spinor fields, the solutions of the type (2) are found to be unstable (Rybakov, 1965, 1966a). From this point of view the models with topological charges are perspective (Skyrme, 1962).

REFERENCES

Bialynicki–Birula, I. and Mycielski, J. (1975). *Bulletin de l'Académie Polonaise des Sciences, Classe* 3, **23**, 467.

Coleman, S., Glaser, V., and Martin, A. (1978). *Communications in Mathematical Physics,* **58**, 211.

Courant, R. and Hilbert, D. (1953). *Methods of Mathematical Physics,* Vol. 1, Chap. VI. Interscience Publishers, New York.

Faddeev, L. D. and Korepin, V. E. (1978). *Physics Reports,* **42**, 1.

Friedberg, R., Lee, T. D., and Sirlin, A. (1976). *Physical Review,* D **13**, 2739.

Jidkov, E. P. and Shirikov, V. P. (1964). *Journal of Computational Mathematics and Mathematical Physics,* **4**, 804 (in Russian).

Makhankov, V. G. (1978). *Physics Reports,* **35**, 1.

Movchan, A. A. (1960). *Applied Mathematics and Mechanics,* **24**, 988 (in Russian).

Rybakov, Yu. P. (1965). Messenger of the Moscow State University, Ser. III, No. 6, 62 (in Russian).

Rybakov, Yu. P. (1966a). Messenger of the Moscow State University, Ser. III, No. 1, 72 (in Russian).

Rybakov, Yu. P. (1966b). In *Problems of Gravitation and Elementary Particles Theory,* K. P. Staniukovich, ed. p. 68. Atomizdat, Moscow (in Russian).

Rybakov, Yu. P. (1978). In *Problems of Statistical and Quantum Physics,* p. 91. P. Lumumba University, Moscow (in Russian).

Skyrme, T. H. R. (1962). *Nuclear Physics,* **31**, 556.

Slobodkin, A. M. (1964). *Proceedings of the U.S.S.R. Academy of Sciences,* **157**, 63 (in Russian).

Synge, J. L. (1961). *Proceedings of the Royal Irish Academy of Sciences,* **62A**, 17.

Tricomi, F. G. (1961). *Differential Equations,* Chap. III. Blackie and Son Ltd., London.

Vakhitov, N. G. and Kolokolov, A. A. (1973). *Izvestiya Vuzov, Radiophysics,* **16**, 1020 (in Russian).

Zastavenko, L. G. (1965). *Applied Mathematics and Mechanics,* **29**, 430 (in Russian).

Zubov, V. I. (1957). *Lyapunov's Methods and their Applications.* Leningrad State University, Leningrad.